ON A FAMILY OF CIRCLE HOMEOMOHPHISMS WITH ONE BREAK POINT

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Abstract. In this article, we study a one-parameter family of circle homeomorphisms with one break point. It is proved that in the case of a rational rotation number the number of periodic trajectories does not exceed two.

Key words: circle homeomorphism, renormalization, rotation number.

О СЕМЕЙСТВЕ КРУГЛЫХ ГОМЕОФИЗМОВ С ОДНИМ ТОЧКОЙ РАЗЛОМА

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Аннотация. В этой статье мы изучаем однопараметрическое семейство гомеоморфизмов окружности с одной точкой излома. Доказано, что в случае рационального числа вращения число периодических траекторий не превышает двух.

Ключевые слова: гомеоморфизм окружности, перенормировка, число вращения.

Consider a one-parameter family of mappings of the unit circle [1]:

$$T_{\Omega}x = \{f(x, \Omega)\}, \quad x \in S^1 = [0, 1), \ \Omega \in [0; 1]$$

where the bracket $\{\}$ - denotes the fractional part of the number, and $f(x, \Omega)$ - satisfies the following conditions:

$a)$ at a fixed Ω, $f(x; \Omega)$ - continuous monotonically increasing function;

$b)\quad f(0;0)=0,\ f(x+1;\Omega)=f(x;\Omega)+1,$ for anyone $x\in R^1$;

$c)\quad \frac{\partial f(x;\Omega)}{\partial\Omega}>const>0;$

$d)\ t_0:[0;1]\to[0;1]$ continuous curve;

$e)$ for every fixed $\Omega\in[0;1],\ \frac{\partial f(x;\Omega)}{\partial x}>const>0;$ for $\forall x\in S^1\setminus\{t_0(\Omega)\},\ f(x;\Omega)\in C^{2+\varepsilon}(S^1\setminus\{t_0(\Omega)\})$, at some $\varepsilon>0$ and

$$\frac{f_-^{'}(t_0(\Omega),\Omega)}{f_+^{'}(t_0(\Omega),\Omega)}=c(\Omega)\neq 1.$$

Let us ρ_Ω denote the number of rotations corresponding to, responsible T_{f_Ω} [2]:

$$\rho_\Omega=\lim_{n\to\infty}\frac{f^{(n)}(x,\Omega)}{n}$$

From the $d)-e)$ conditions it follows that $T_\Omega x$ for each fixed value of the parameter has only one break point $t_0(\Omega)$. The number $c(\Omega)$ is called the break point T_Ω. Everywhere below we will denote by the $f^{(n)}-n$ st superposition of the function f. It is easy to see that ρ_Ω monotonically (not strictly) depends on the parameter Ω. Note that each rational $\rho=\frac{p}{q}$ corresponds to a non-degenerate segment (values Ω such that $\rho_\Omega=\frac{p}{q}$, while irrational ρ corresponds only to Ω).

Let $A=\left(\frac{p_1}{q_1},\frac{p_2}{q_2}\right)\subset(0,1)$ - be the Faria interval of the $n-$nd level [1]:

1) $p_2q_1-p_1q_2=1$

2) All rational numbers inside the interval A have the form $\frac{kp_1+lp_2}{kq_1+lq_2}$. Rational number with minimum denominator is $\frac{p_1+p_2}{q_1+q_2}$.

We choose an arbitrary point x_0 on the circle and a segment of the trajectory of this point $\{x_i=T_\Omega^i x_0, 0\le i<q_1+q_2\}$. Denote $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ segments $[x_0,x_{q_1}]$ and $[x_{q_2},x_0]$, respectively. We also denote the images of these segments under the action of T_Ω by $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ [1]:

$$\Delta_i^{(1)}=T^i\Delta_0^{(1)},\quad \Delta_j^{(2)}=T^i\Delta_0^{(2)}\quad .$$

The following assertion was proved in [1] and works without any changes in our situation. **Лемма 1.** Suppose $\rho(T)\in\left(\frac{p_1}{q_1},\frac{p_2}{q_2}\right)$. Trajectory segment $\{x_i=T_\Omega^i x_0, 0\le i<q_1+q_2\}$ divides the circle into non-intersecting segments $\Delta_i^{(1)},\ 0\le i<q_2$ and $\Delta_j^{(2)},\ 0\le j<q_1$. Denote the constructed partition $\xi(A,x_0)$. Let's put $\upsilon=\upsilon ar_{S^1}\ln f'<\infty, \bar{\upsilon}=\upsilon+|\ln f'(x_0-0)+\ln f'(x_0+0)|$, $q=\max(q_1,q_2),\ p=\max(p_1,p_2)$. Consider an arbitrary trajectory $y_i=T_\Omega^i y_0,\ y_0\in S^1$ such that $y_i\ne x_0=0,\ 0\le i<q_2$.

Лемма 2. Suppose $\rho(T)\in\left(\frac{p_1+p_2}{q_1+q_2},\frac{p}{q}\right)$ or $\rho(T)=\left(\frac{p}{q}\right)$. Then

$$e^{-\bar{\upsilon}}\le\prod_{i=0}^{q-1}f'(y_i)\le e^{\bar{\upsilon}}.$$

Let $A_n=\left(\frac{p_1}{q_1},\frac{p_2}{q_2}\right)$ be a Farey interval of rank $n-$ [1], and $A_m, m<n$ - be some Farey interval of rank m containing A_n. Let $\rho(T)\in A_n$. Let's choose Δ_n- arbitrary element of partition $\xi(A_m,t_0)$ containing Δ_n. Let's denote by $|\Delta|$.

Лемма 3. Let's put $\lambda=(1+e^{-\bar{\upsilon}})^{-\frac{1}{2}}<1$.

$$|\Delta_n|\le const\lambda^{n-m}|\Delta_m|,\ \ |\Delta_n|\le const\lambda^n.$$

Let the continued fraction expansion of ρ be of the form $\rho(f(x,\Omega))=\frac{p}{q}=[k_1,k_2,\dots,k_n],k_n\ge 2.$

Let's designate $I(\frac{p}{q})$ the segment of the value of the parameter Ω such that $\rho(\Omega)=\frac{p}{q}$. Fix some $\Omega\in I(\frac{p}{q})$ and denote $f=f_\Omega,\ T_f=T_{f_\Omega}$. For a rational rotation number of $\rho(\Omega)=\frac{p}{q}$, there always exists at least one periodic trajectory of period q. Let $\{y^{(i)},0\le i\le q-1\}$ be an arbitrary periodic trajectory. Let $[y_1,y_2]$ denote the segment formed by the trajectory $\{y^{(i)},0\le i\le q-1\}$ and containing the singular point t_0. Let's move on to renormalized coordinates:

$$x=y_2+(y_1-y_2)z$$

and define the function corresponding to T_f^q in the renormalized coordinate system:

$$\overline{f}(z) = \frac{1}{y_1 - y_2}[T_f^q(y_2 + (y_1 - y_2 z)) - y_2],\ z \in [0, 1].$$

Denote by d the renormalized coordinate of the point t_0:

$$d = (t_0 - y_2)/(y_1 - y_2)$$

and define the function $F_d(z),\ z \in [0, 1]$:

$$F_d(z) = \begin{cases} \dfrac{zc}{d(1-c^2) + c^2 + z(c-1)}, & z \in [0, d] \\ \dfrac{d(1-c^2) + zc^2}{d(1-c^2) + c + zc(c-1)}, & z \in [d, 1] \end{cases}$$

Theorem 1. There is a constant $c_3 > 0$ such that

$$\left\| \overline{f}(z) - F_d(z) \right\|_{C^2([0,1]\setminus\{d\})} \le c_3 \lambda^{n\varepsilon}. \qquad (1)$$

Proof. Consider the partition of the circle generated by the trajectory $(y^{(i)},\ 0 \le i \le q-1)$. Denote $\Delta_0 = [y_1, y_2],\ \Delta_i = T_\theta^i \Delta_0,\ 1 \le i \le q-1$. Obviously $T_\theta^q \Delta_0 = \Delta_0$. It is not difficult to show [1] that $|\Delta_i| \le const\,\lambda^n,\ 1 \le i \le q-1$. Function $\overline{f}(z)$ can be represented as a superposition of two functions f_1 and f_2, corresponding to mappings $T_\theta : \Delta_0 \to \Delta_1,\ T_\theta^{q-1} : \Delta_1 \to \Delta_q = \Delta_0$. Let us determine the relative coordinates inside the segments Δ_i:

$$x = T_f^i y_2 + (T_f^i y_1 - T_f^i y_2) z.$$

Then the functions f_1 and f_2 can be written as:

$$f_1(z_0) = \frac{1}{(T_\theta y_1 - T_\theta y_2)}[T_\theta(y_2 + (y_1 - y_2)z_0) - T_\theta y_2]$$

$$f_2(z_1) = \frac{1}{y_1 - y_2}[T_\theta^{q-1}(T_\theta y_2 + (T_\theta y_1 - T_\theta y_2)z_1) - y_2]$$

Wherein

$$\overline{f}(z) = f_2(f_1(z)). \qquad (2)$$

In [1], it was proved

$$\left\| f_2(z_1) - \frac{M z_1}{1 + z_1(M-1)} \right\|_{C^2([0,1])} \le const\,\lambda^{n\varepsilon} \qquad (3)$$

where $\ln M=\sum_{i=1}^{q-1}\int_{\Delta_i}\frac{f''(y)}{2f'(y)}dy=\left(\sum_{i=0}^{q-1}\int_{\Delta_i}\frac{f''(y)}{2f'(y)}dy\right)-\int_{\Delta_0}\frac{f''(y)}{2f'(y)}dy=\ln c-\int_{\Delta_0}\frac{f''(y)}{2f'(y)}dy$ (4)

Insofar as $\left|\int_{\Delta_0}\frac{f''(y)}{2f'(y)}dy\right|\le const\ \lambda^n$, we get

$$\left\|f_2(z_1)-\frac{cz_1}{1+z_1(c-1)}\right\|_{C^2([0,1])}\le const\ \lambda^{n\varepsilon} \qquad (5)$$

It is easy to see that function $f_1(z_0)$ is close to piecewise linear function $f_d(z_0)$, where

$$f_d(z_0)=\begin{cases}\dfrac{z_0}{c^2(1-d)+d}, & z_0\in[0,d]\\ \dfrac{d(1-c^2)+z_0c^2}{c^2(1-d)+d}, & z_0\in[d,1]\end{cases} \qquad (6)$$

Since $|\Delta_0|\le const\ \lambda^n$ is valid estimate:

Using (2)-(6) we obtain (1).

Theorem 1 implies that $\overline{f}(z)$ is convex at $0<c<1$ and concave at $c>1$. Indeed, by direct calculation it is easy to verify that

$$\frac{d^2}{dz^2}F_d(z)\ge 2c^2(1-c),\ z\ne d \quad \text{при } 0<c<1$$

$$\frac{d^2}{dz^2}F_d(z)\le -\frac{2}{c^3}(c-1),\ z\ne d \quad \text{при } c>1.$$

Let's put

$$N=\left[\frac{1}{\varepsilon\ln\lambda}\ln\left(\frac{1}{c^3}|c-1|\min\left(\frac{1}{c^3},c^2\right)\right)\right].$$

Denote the interval $I(\frac{p}{q})=\left[\Omega_1(\frac{p}{q}),\Omega_2(\frac{p}{q})\right]$.

Let's put $J=[0,1]\setminus\bigcup_{0\le\frac{p}{q}\le 1}I(\frac{p}{q})$. Denote the Lebesgue measure on $[0,1]$ by λ

.

We now formulate the main results of our work.

Theorem 2. For all $n>N$, the following statements are true:

(a) if $\Omega=\Omega_1(\frac{p}{q})$ or $\Omega=\Omega_2(\frac{p}{q})$, then T_Ω has a unique periodic trajectory of period q;

(c) at $\Omega\in\left(\Omega_1(\frac{p}{q}),\Omega_2(\frac{p}{q})\right)$ there are equal to two periodic trajectories of period q.

Theorem 3. The Lebesgue measure of the set J is equal to zero, i.e. $\lambda(J)=0$.

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